

Instanton on toric singularities and black hole countings

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ABSTRACT: We compute the instanton partition function for $\mathcal{N} = 4$ U(N) gauge theories living on toric varieties, mainly of type $\mathbb{R}^4/\Gamma_{p,q}$ including A_{p-1} or $O_{\mathbb{P}^1}(-p)$ surfaces. The results provide microscopic formulas for the partition functions of black holes made out of D4-D2-D0 bound states wrapping four-dimensional toric varieties inside a Calabi-Yau. The partition function gets contributions from regular and fractional instantons. Regular instantons are described in terms of symmetric products of the four-dimensional variety. Fractional instantons are built out of elementary self-dual connections with no moduli carrying non-trivial fluxes along the exceptional cycles of the variety. The fractional instanton contribution agrees with recent results based on 2d SYM analysis. The partition function, in the large charge limit, reproduces the supergravity macroscopic formulae for the D4-D2-D0 black hole entropy.

KEYWORDS: Nonperturbative Effects, Supersymmetric gauge theory, Black Holes in String Theory, Solitons Monopoles and Instantons.

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1. Introduction

Recent studies in string theory have revealed remarkable connections between apparently unrelated subjects like black hole thermodynamics, topological strings and instanton physics. Not surprisingly, the key role in the game is played by D-branes. The typical example being type IIA string on a Calabi-Yau (CY) threefold. A black hole in the resulting $\mathcal{N} = 2$ four-dimensional supergravity can be built out of D0-D2-D4-D6 bound states wrapping various cycles inside the CY. The black hole partition function Z_{BH} can be defined as the thermal partition function with (canonical) microcanonical ensemble for (electric) magnetic charges (Q_0, \vec{Q}_2) \vec{Q}_4, Q_6 :

$$Z_{BH} = \sum_{Q_0, \vec{Q}_2} \Omega(Q_0, \vec{Q}_2, \vec{Q}_4, Q_6) e^{-Q_0 \varphi_0 - \vec{Q}_2 \cdot \vec{\varphi}_2} \tag{1.1}$$

with $\Omega(Q_0, \vec{Q}_2, \vec{Q}_4, Q_6)$ counting the number of D0-D2-D4-D6 bound states. We will always take $Q_6 = 0$ (no D6-branes) and write $\vec{Q}_4 = N$ to denote N D4-branes wrapping a given four-cycle M inside the CY. For large N , Z_{BH} can be computed in supergravity as the

exponential of the microcanonical black hole entropy. The result can be written in the suggestive form [1–3]

$$Z_{\text{BH}} = |Z_{\text{top}}|^2 + \dots \tag{1.2}$$

where Z_{top} is the partition function of the topological string on the CY and the dots stand for $O(e^{-N})$ corrections.

On the other hand a D0-brane can be thought of as an instanton on the twisted $\mathcal{N} = 4$ supersymmetric four-dimensional gauge theory (SYM) living on the D4-brane. In the same spirit, D2-branes are associated to magnetic fluxes along the D4-worldvolume, i.e. to non-trivial first Chern classes of the instanton gauge bundle. When the D4-branes wrap an A_{p-1} ALE singularity, the moduli space of the D0-D2-D4 system can be realized by the ADHM construction [4]. The multiplicities of the D0-D2-D4 bound states are given by the number of vacua (Witten index) of the gauge theory living at the zero dimensional brane intersection. This is a quantum mechanics with target space the ADHM manifold. Since coordinates and differentials on the ADHM instanton moduli space can be interpreted as bosonic and fermionic degrees of freedom of the quantum mechanics, the Witten index gives the Euler number of the instanton moduli space. The generating function for the Euler character of the instanton moduli spaces gives the SYM partition function, Z_{SYM} , of the $\mathcal{N} = 4$ theory. This leads to an alternative route to compute the black hole partition function

$$Z_{\text{BH}} = Z_{\text{SYM}} \tag{1.3}$$

The relation (1.2) becomes then a highly non-trivial statement about the gauge theory: Z_{SYM} should factorize at large N . This factorization has been confirmed by explicit computations in [5–9] where the study of the $d = 4$ SYM partition function has been addressed via 2d-SYM techniques. More precisely, the authors study D4-branes wrapping a $O_{\Sigma_g}(-p)$ four-cycle inside the non-compact CY $O(p - 2 + 2g) \oplus O(-p) \rightarrow \Sigma_g$. The computation was performed in a q -deformed 2d SYM theory following from dimensional reduction of $\mathcal{N} = 4$ SYM down to Σ_g . The partition function of the q -deformed 2d SYM theory was written as a sum over $U(N)$ representations and the factorization was proved to hold in the large N limit. After a Poisson resummation the results were cast in a form that resembles a four-dimensional instanton sum.

In this paper we compute the instanton partition function directly from the four dimensional perspective. The study of instantons effects in four-dimensional gauge theories has been considered for a long time an "out of reach" task due to the highly non-trivial structure of the ADHM instanton moduli spaces. This situation drastically changed with the discovery [10] that the instanton partition function (and chiral SYM amplitudes) localize around a finite number of critical points in the ADHM manifold. This leads to an impressive number of results in the study of multi-instanton corrections to $\mathcal{N} = 1, 2, 4$ supersymmetric gauge theories in \mathbb{R}^4 [11–18] (see [19] for a review of multi-instanton techniques before localization and a complete list of references).

Here we apply the localization techniques to the study of instantons on toric varieties coming from blowing up orbifold singularities of the type of $M_4 = \mathbb{C}^2/\Gamma_{p,q}$. $\Gamma_{p,q}$ denotes a \mathbb{Z}_p -action specified by the pair of coprime integers (p, q) . This includes the A_{p-1} singu-

larities and the blown down $O_{\mathbb{P}^1}(-p)$ surfaces. The general (p, q) case describes the most general toric singularity in four dimensions. The ADHM construction of instantons on a A_{p-1} -singularity was carried out in [4]. In [20] this construction was applied to the study of the prepotential of $\mathcal{N} = 2$ $SU(N)$ SYM theories on the ALE space (see also [21]). The results were developed and put in firm mathematical grounds in [22]. Here we generalize these results to the case of D4-D2-D0 bound states on general toric varieties. We work in details the case of $\mathbb{C}^2/\Gamma_{p,q}$ singularities. We start by revisiting the case of instantons on A_{p-1} -spaces. We give a complete description of the gauge bundle and present an alternative derivation of the instanton partition function which naturally extends to instantons on a general (compact or not) toric variety where an explicit ADHM construction is not known. The resulting instanton formula will be tested against 2d SYM results and supergravity macroscopic black hole entropy formulae that suggest that our results apply to general (compact or not) toric varieties at least in the limit of large instanton charges. The instanton partition functions will be written in terms of modular invariant forms consistently with the $SL(2)$ -invariance of the $\mathcal{N} = 4$ gauge theory.

The paper is organized as follows: In section 2, 3 we review and elaborate on the ADHM construction of instanton moduli spaces on \mathbb{C}^2 and A_{p-1} . We give a detail description of the instanton Chern classes and compute the SYM partition function. Sections 4 deals with instantons on a general $\mathbb{C}^2/\Gamma_{p,q}$ singularity. In section 5, we derive the microscopic D4-D2-D0 black hole partition function and test it against supergravity. Section 6 summarizes our results. In appendix A we review the regular/fractional factorization algorithm developed in [22]. In appendix B we collect some useful background material in toric geometry.

2. ADHM on \mathbb{C}^2

The moduli space of self-dual $U(N)$ connections on \mathbb{C}^2 can be described as a $U(k)$ quotient of a hypersurface on \mathbb{C}^{2k^2+2kN} defined by the ADHM constraints

$$\begin{aligned} [B_1, B_2] + IJ &= 0 \\ [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J &= \xi \mathbf{1}_{k \times k}. \end{aligned} \tag{2.1}$$

The coordinates in \mathbb{C}^{2k^2+2kN} are represented by $B_{1,2}$, I and J given by $[k] \times [k]$, $[k] \times [N]$ and $[N] \times [k]$ dimensional matrices respectively. The $U(k)$ action is defined as

$$B_\ell \rightarrow U B_\ell U^\dagger \quad I \rightarrow U I \quad J \rightarrow J U^\dagger \tag{2.2}$$

with U a $[k] \times [k]$ matrix of $U(k)$. There is a nice D-brane description of this system. A $U(N)$ instanton with instanton number k can be thought as a bound state of k $D(-1)$ and N $D3$ -branes. The instanton moduli B_ℓ, I, J represent the lowest modes of open strings connecting the various branes. The ADHM constraints are identified with the F- and D-term flatness conditions in the effective 0-dimensional theory. Finally ξ , a Fayet-Iliopoulos term, measures the non-commutativity of spacetime and is needed in order to regularize the moduli space. All physical quantities we will deal with are independent of the value of ξ and therefore we can think of it as a regularization artifact.

The tangent space matrices $\delta B_{1,2}, \delta I, \delta J$ can be thought of as the homomorphisms

$$\begin{aligned}\delta B_\ell &\in V \otimes V^* \otimes Q \\ \delta I &\in V \otimes W^* \\ \delta J &\in W \otimes V^* \otimes \Lambda^2 Q\end{aligned}\tag{2.3}$$

between the spaces V, W, Q with dimensions $[k], [N]$ and $[2]$ respectively. In the brane picture V, W parametrize the space of D(-1) and D3 boundaries while Q is a doublet respect to a $SU(2)$ subgroup of the Lorentz group. For the sake of simplicity from now on we will omit the tensor product symbol and use $+$ instead of \oplus . Product of spaces will be always understood as tensor products.

The tangent instanton moduli space can then be written as [25]

$$T\mathcal{M}_k = V^* V [Q - \Lambda^2 Q - 1] + W^* V + V^* W \Lambda^2 Q\tag{2.4}$$

The contributions with the minus signs come from the three ADHM constraints (2.1) and the $U(k)$ invariance ($2k^2$ complex degrees of freedom in total). The dimension of the moduli space \mathcal{M}_k can be easily read from (2.4) to be $\dim_{\mathbb{C}} \mathcal{M}_k = k^2(2 - 2) + 2kN = 2kN$.

SYM amplitudes involve integrals over the ADHM manifold. For chiral correlators these integrals localize around isolated fixed points of the instanton symmetry group $U(k) \times U(N) \times SO(4)$ on the ADHM manifold [10]. Moreover for $\mathcal{N} = 4$ SYM the contribution of the vector supermultiplet cancels against that of the hypermultiplet and the instanton partition function reduces to the counting of the number of fixed points, i.e. the Euler character of the instanton moduli space [12]. In this paper we will mainly deal with the evaluation of this index.

The spaces V, W transform in the fundamental representation of $U(k)$ and $U(N)$ respectively while Q transforms in the chiral spin representation of $SO(4)$. We parametrize the action of the Cartan symmetry group $U(1)^{N+k+2}$ by $a_\alpha, \phi_s, \epsilon_l$ with $\alpha = 1, \dots, N$, $s = 1, \dots, k, l = 1, 2$. Fixed points are in one to one correspondence with sets of N Young tableaux $Y = (Y_1, \dots, Y_N)$ with $k = \sum_\alpha k_\alpha$ boxes distributed between the Y_α 's. The boxes in a Y_α diagram are labelled by the index $s = (\alpha, i_\alpha, j_\alpha)$ with i_α, j_α denoting the horizontal and vertical position respectively in the Young diagram Y_α . More precisely

$$e^{i\phi_s} = e_\alpha T_1^{-j_\alpha+1} T_2^{-i_\alpha+1}\tag{2.5}$$

with $T_\ell = e^{i\epsilon_\ell}$, $e_\alpha = e^{ia_\alpha}$. The tangent instanton moduli space is spanned by the fluctuations $\delta B_\ell, \delta I, \delta J$ satisfying the linearized ADHM constraints around the fixed point. Chiral SYM amplitudes can be related to the character of the $U(1)^{k+N+2}$ action evaluated at the fixed points on the tangent space. Although we are mainly interested here on the $\mathcal{N} = 4$ SYM partition given by the blind counting of fixed points, for future references we will write (when possible) explicit expressions for the full $U(1)^{k+N+2}$ character.

The character of a space \mathcal{H} will be defined as follows

$$\chi_\epsilon(\mathcal{H}) \equiv \text{Tr}_{\mathcal{H}} e^{ia_\alpha J^\alpha + i\phi_s J^s + i\epsilon_l J^l}\tag{2.6}$$

with J^α, J^s, J^l the generators of the Cartan group $U(1)^{k+N+2}$. In particular at a given fixed point $Y = \{Y_\alpha\}$ one finds

$$\begin{aligned} \chi_\epsilon(V) &= \sum_{(\alpha, i_\alpha, j_\alpha) \in Y_\alpha} e_\alpha T_1^{1-j_\alpha} T_2^{1-i_\alpha} \\ \chi_\epsilon(W) &= \sum_{\alpha=1}^N e_\alpha \\ \chi_\epsilon(Q) &= T_1 + T_2; \quad Q = Q_1 + Q_2 \end{aligned} \tag{2.7}$$

Plugging (2.7) into (2.4) one finds the character of the instanton moduli space \mathcal{M}_Y at the fixed point Y [11, 12]

$$\chi_\epsilon(\mathcal{M}_Y) = \sum_{\alpha, \beta}^N \mathcal{N}_{\alpha, \beta}^Y(T_1, T_2) \tag{2.8}$$

with

$$\mathcal{N}_{\alpha, \beta}^Y(T_1, T_2) = e_\alpha e_\beta^{-1} \left(\sum_{s_\alpha \in Y_\alpha} T_1^{-h_\alpha(s_\alpha)} T_2^{v_\beta(s_\alpha)+1} + \sum_{s_\beta \in Y_\beta} T_1^{h_\beta(s_\beta)+1} T_2^{-v_\alpha(s_\beta)} \right) \tag{2.9}$$

and

$$\begin{aligned} h_\alpha(s_\alpha) &= h_\alpha(i_\alpha, j_\alpha) = \nu_{j_\alpha}^{Y_\alpha} - i_\alpha \\ v_\alpha(s_\beta) &= v_\alpha(i_\beta, j_\beta) = \tilde{\nu}_{i_\beta}^{Y_\alpha} - j_\beta \end{aligned} \tag{2.10}$$

Here $\nu_{i_\alpha}^{Y_\alpha} (\tilde{\nu}_{j_\alpha}^{Y_\alpha})$ denotes the lengths of the columns(rows) in the tableau Y_α . In other words $h_\alpha(s_\beta) (v_\alpha(s_\beta))$ denotes in general the number of horizontal (vertical) boxes in the tableau Y_α to the right(on top) of the box $s_\beta \in Y_\beta$. Notice in particular that $h_\alpha(s_\beta)$ is negative if s_β is outside of the tableau Y_α .

3. Instantons on A_{p-1}

In this section we review and elaborate on the ADHM construction of instantons on a A_{p-1} -singularity [4, 20, 22]. The ALE space A_{p-1} is defined by the quotient \mathbb{C}^2/Γ with Γ the \mathbb{Z}_p -action with generator

$$\Gamma : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \rightarrow \begin{pmatrix} e^{2\pi i/p} & 0 \\ 0 & e^{-2\pi i/p} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{3.1}$$

The moduli space of instantons on \mathbb{C}^2/Γ can be found from that on \mathbb{C}^2 by projecting onto its Γ -invariant component. In this projection, the fixed points under the action of $U(1)^{N+k+2}$ on the \mathbb{C}^2 -ADHM moduli space are preserved since $\Gamma \in U(1)^{N+k+2}$. However, the projection splits the \mathbb{C}^2 instanton moduli into several disjoint pieces classified by the choice of the \mathbb{Z}_p representations under which the D(-1) instantons and D3-branes transform. Fixed points are now described by p-coloured Young tableaux. More precisely, $U(N)$

instantons on \mathbb{C}^2/Γ are specified by the N sets $\{(Y_\alpha, r_\alpha)\}$ with Y_α a tableau with p types of boxes and r_α an integer *mod* p . The label r_α specifies the \mathbb{Z}_p representation R_{r_α} under which the first box in Y_α transforms. The p choices correspond to the p types of fractional D3-branes. More precisely the r_α 's specify the embedding of Γ into the $U(1)^{N+2}$ symmetry group

$$\Gamma : \quad R_a \rightarrow e^{2\pi i a/p} R_a \quad T_1 \rightarrow e^{\frac{2\pi i}{p}} T_1 \quad T_2 \rightarrow e^{-\frac{2\pi i}{p}} T_2 \quad e_\alpha \rightarrow e^{\frac{2\pi i r_\alpha}{p}} e_\alpha \quad (3.2)$$

Notice that this action induces also an action on V via (2.5). Indeed the box (i_α, j_α) of the tableaux Y_α transforms in the representation $R_{r_\alpha+i_\alpha-j_\alpha}$. The spaces V, W decompose as

$$\begin{aligned} V &= \sum_{a=0}^{p-1} V_a R_a; & \dim V_a &= k_a \\ W &= \sum_{a=0}^{p-1} W_a R_a; & \dim W_a &= N_a \end{aligned} \quad (3.3)$$

with k_a, N_a counting the number of times the a^{th} -representation appears in V and W respectively. Notice that N_a is specified by the choice of r_α via $N_a = \sum_\alpha \delta_{r_\alpha, a}$. The unbroken symmetry group is $SU(2) \times U(1) \times \prod_{a=0}^{p-1} U(N_a) \times U(k_a)$ with $SU(2) \times U(1)$ the isometry of the ALE space.

The tangent of the instanton moduli space is then given by the Γ -invariant component of the \mathbb{C}^2 result [20]

$$\begin{aligned} T\mathcal{M}_Y &= (V^* V [Q - \Lambda^2 Q - 1] + W^* V + V^* W \Lambda^2 Q)^\Gamma \\ &= \sum_a (V_a [V_{a+1}^* Q_1 + V_{a-1}^* Q_2 - V_a^* - V_a^* Q_1 Q_2] + V_a W_a^* + V_a^* W_a Q_1 Q_2) \end{aligned} \quad (3.4)$$

Here and below the subscript a will be always understood *mod* p . The dimension of the moduli space is given by

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{M}_Y &= (k_a k_{a+1} + k_a k_{a-1} - 2k_a^2 + 2k_a N_a) \\ &= -\widehat{C}_{ab} k_a k_b + 2k_a N_a \end{aligned} \quad (3.5)$$

with $\widehat{C}_{ab} = 2\delta_{ab} - \delta_{a,b+1} - \delta_{a,b-1}$ the extended A_{p-1} Cartan matrix. Repeated indices are understood to be summed over $a, b = 0, \dots, p-1$. Notice that the complex dimension is always even, in agreement with the fact that the instanton moduli space on A_{p-1} is hyperkähler.

The character is given by the Γ invariant components of (2.9)

$$\chi_\epsilon(\mathcal{M}_Y) = \sum_{\alpha, \beta}^N \mathcal{N}_{\alpha, \beta}^Y(T_1, T_2)^\Gamma \quad (3.6)$$

with $\mathcal{N}_{\alpha, \beta}^Y(T_1, T_2)^\Gamma$ the restriction of the \mathbb{C}^2 result to those monomials invariant under (3.2).

Now let us describe the instanton gauge bundle. The A_{p-1} singularity can be resolved by the blowing up procedure which consists in replacing the singularity with $p-1$ intersecting spheres, \mathbb{P}^1 (called exceptional divisors). With respect to the \mathbb{R}^4 case, this leads to new

self-dual connections with non-trivial fluxes along the exceptional divisors. The instanton bundle can then be constructed out of elementary $U(1)$ bundles, \mathcal{T}^a , $a = 0, \dots, p-1$, carrying the unit of flux through the exceptional divisors. \mathcal{T}^0 denotes the trivial bundle. Following [23] we will refer to this bundle as the tautological bundle. In the ADHM construction the $U(1)$ bundle \mathcal{T}^a , corresponds to a tableau with no boxes $k_1 = k_2 = \dots = 0$ and $r = a$.

The $p-1$ non-trivial bundles \mathcal{T}^a with $a = 1, \dots, p-1$ can be used as a basis for two forms on A_{p-1} with intersection matrix

$$\int c_1(\mathcal{T}^a) \wedge c_1(\mathcal{T}^b) = -C^{ab} \quad \int_{C_a} c_1(\mathcal{T}^b) = \delta_a^b \quad (3.7)$$

C^{ab} , being the inverse of the A_{p-1} -Cartan matrix. Here and below, for the sake of simplicity, we have extended the range of the a -index to $a = 0, \dots, p-1$ by defining $C^{0a} = 0$. The gauge bundle F_Y is given by [4]

$$\begin{aligned} F_Y &= (V^* \mathcal{T} [Q - \Lambda^2 Q - 1] + W^* \mathcal{T})^\Gamma \\ &= \sum_a \mathcal{T}^a (V_{a+1}^* Q_1 + V_{a-1}^* Q_2 - V_a^* Q_1 Q_2 - V_a^* + W_a^*) \end{aligned} \quad (3.8)$$

with $\mathcal{T} = \sum_a \mathcal{T}^a R_a$ the tautological bundle.

The Chern characters are given by¹

$$\begin{aligned} \text{ch}_1(F_Y) &= \sum_a u_a \text{ch}_1(\mathcal{T}^a) \\ \text{ch}_2(F_Y) &= \sum_a u_a \text{ch}_2(\mathcal{T}^a) - \frac{K}{p} \Omega \quad K = \sum_a k_a \end{aligned} \quad (3.9)$$

with Ω the normalized volume form of the manifold, $\text{ch}_1 = c_1$, $\text{ch}_2 = -c_2 + \frac{1}{2}c_1^2$ and

$$u_a = N_a + k_{a+1} + k_{a-1} - 2k_a = N_a - \widehat{C}_{ab} k_b \quad (3.10)$$

Notice that $\sum_a u_a = N$ therefore u_a 's are characterized by $p-1$ independent components.

The instanton number $k \in \frac{1}{2p}\mathbb{Z}$ is defined as

$$k = - \int_M \text{ch}_2(F_Y) = \frac{1}{2} \sum_a C^{aa} u_a + \frac{K}{p} = k_0 + \frac{1}{2} \sum_a C^{aa} N_a \quad (3.11)$$

with

$$C^{aa} = \frac{1}{p}(p-a)a \quad (3.12)$$

Formula (3.11) shows that the instanton number can be computed by counting the numbers of 0's in the Young tableaux set Y .

The instanton bundle will be labelled then by u_a with $a = 1, 2, \dots, p-1$ and k specifying the first and second Chern characters respectively. We will compute the following index

$$\mathcal{Z}(q, z_a) \equiv \left\langle e^{\varphi_{2a} \int c_1(F) \wedge c_1(\mathcal{T}^a)} \right\rangle = \sum_{k, u_a} \chi(\mathcal{M}_{k, u_a}) q^k e^{-z^a u_a} \quad z^a \equiv C^{ab} \varphi_{2b} \quad (3.13)$$

¹Our conventions: $c(F) = \sum_i c_i(F) t^i = \det(1 + t \frac{iF}{2\pi})$, $\text{ch}(F) = \sum_i \text{ch}_i(F) t^i = \text{tr} e^{t \frac{iF}{2\pi}}$, $F = dA$.

where $q = e^{2\pi i \tau}$, τ is the complexified coupling constant and $\chi(\mathcal{M}_{k,u_a})$ is the Euler number of the instanton moduli space with first and second Chern characters u_a and k respectively. We will refer to $\mathcal{Z}(q, z_a)$ as the instanton partition function. This index counts the number of bound states formed by N D4 branes, u_a D2 branes (wrapping the a exceptional divisor) and k D0 branes. As we explained before $\chi(\mathcal{M}_{k,u_a})$ can be computed by simply counting the number of Young tableaux for a given k, u_a .

The result (3.13) can be refined by introducing the Poincaré polynomial generating function

$$\mathcal{P}(t, q, z_a) \equiv \sum_{u_a, k} P(t|\mathcal{M}_{k,u_a}) q^k e^{-z_a u_a} = \sum_{u_a, k} b_i(\mathcal{M}_{k,u_a}) t^i q^k e^{-z_a u_a} \quad (3.14)$$

with $P(t|\mathcal{M}_{k,u_a})$ the Poincaré polynomial of the instanton moduli space and $b_i(\mathcal{M}_{k,u_a})$ its Betti numbers. Notice that one can specify the fixed point either by k, u_a or by $\{k_a\}$. As shown in [20] each choice of $\{k_a\}$ and therefore of k, u_a leads to a disconnect piece of the instanton moduli space, i.e. $b_0(\mathcal{M}_{k,u_a}) = 1$.

The Betti numbers can be computed by taking

$$T_1 \gg e_1 \gg e_2 \gg \dots e_N \gg T_2 \quad (3.15)$$

and counting the number of negative eigenvalues in the character $\chi_\epsilon(\mathcal{M}_Y)$. More precisely each fixed point can be associated to a harmonic form on the moduli space and therefore contributes once to the Poincaré polynomial. This contribution is given by $t^{2n_-^Y}$, with n_-^Y the number of negative eigenvalues in $\chi_\epsilon(\mathcal{M}_Y)^2$.

Summarizing, fixed points in the moduli space of $U(N)$ instantons on \mathbb{C}^2/Γ are specified by the N sets (Y_α, r_α) , $\alpha = 1, \dots, N$ with Y_α p -colored Young tableaux and $r_\alpha = 0, \dots, p-1$ an integer specifying the \mathbb{Z}_p -representation under which the first box in Y_α transforms. Different choices of $\{r_\alpha\}$ describe in general different breakings $\prod_{a=0}^{p-1} U(N_a) \times U(k_a)$ of the starting $U(N) \times U(K)$ symmetry group. The tangent space is given by projecting the \mathbb{C}^2 character under (3.2). The instanton gauge bundle is given by (3.8) with Chern characters (3.9) and (3.11). The instanton partition function is defined in (3.13) and can be computed by counting the number of Young tableaux with fixed Chern characteristics.

3.1 Regular vs fractional instantons

For ALE spaces there are two types of instantons: regular and fractional ones. A regular instanton is an instanton in the regular representation of $\Gamma = \mathbb{Z}_p$. This type of instanton is free to move (together with its images) on $\mathbb{C}^2/\mathbb{Z}_p$. The moduli space of $K = kp$ regular instantons is then given by choosing symmetrically k points on $\mathbb{C}^2/\mathbb{Z}_p$, i.e. $\mathcal{M}_{kp}^{\text{reg}} = (\mathbb{C}^2/\mathbb{Z}_p)^k/S_k \sim (\mathbb{C}^2/\mathbb{Z}_p)^{[k]}$, with $M^{[k]}$ the Hilbert scheme of k -points on M . Fractional instantons on the other hand correspond to instantons with no moduli associated to positions in the four-dimensional space, i.e. no Γ -invariant massless excitation of a string starting and ending on the same D(-1)-brane. Fractional instantons, unlike the

²We have checked that the Poincaré polynomial coming from (3.6) agrees against the results of [20, 24].

regular ones, have no complete images under the action of \mathbb{Z}_p and therefore they cannot move away from the singularity.

We will start by considering the $U(1)$ case. The two main instanton classes are defined by the conditions

$$\begin{aligned} \text{Regular :} & \quad k_0 = k_1 = \dots = k_{p-1} \quad r = 0 \\ \text{Fractional :} & \quad \dim_{\mathbb{C}} \mathcal{M}_Y^{U(1)} = 0 \end{aligned} \quad (3.16)$$

For the sake of simplification in this $U(1)$ case we drop the subscript for the integer r_α . It is important to notice that according to (3.10) regular instantons satisfy $u_0 = N$, $u_1 = u_2 = \dots = 0$, i.e. regular instantons carry always zero first Chern class. Instantons on A_{p-1} can be built out of regular and fractional ones [20, 22]. More precisely the instanton partition function factorizes into a product of a contribution coming from regular and fractional instantons

$$Z_{A_{p-1}} = Z_{\text{frac}} Z_{\text{reg}} \quad (3.17)$$

This can be seen by noticing that the \mathbb{C}^2 partition function can be rewritten as

$$\begin{aligned} Z_{\mathbb{C}^2/\mathbb{Z}_2} &= \bullet_0 + \boxed{0} + \boxed{\begin{array}{c} 1 \\ 0 \end{array}} + \boxed{\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}} + \boxed{\begin{array}{c} 0 \\ 1 \\ 1 \\ 0 \end{array}} + \boxed{\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}} + \boxed{\begin{array}{ccc} 0 & 1 & 1 \\ 0 & 1 & 0 \end{array}} + \boxed{\begin{array}{c} 0 \\ 1 \\ 0 \\ 1 \end{array}} + \boxed{\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}} + \boxed{\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}} + \boxed{\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}} + \boxed{\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}} + \dots \\ &= \left(\bullet_0 + \boxed{0} + \boxed{\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}} + \boxed{\begin{array}{c} 0 \\ 1 \\ 0 \\ 1 \end{array}} + \dots \right) \\ &\quad \times \left(\bullet_0 + \boxed{\begin{array}{c} 1 \\ 0 \end{array}} + \boxed{\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}} + \boxed{\begin{array}{c} 0 \\ 1 \\ 1 \\ 0 \end{array}} + \boxed{\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}} + \boxed{\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}} + \boxed{\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}} + \boxed{\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}} + \dots \right) \\ &= (1 + q_0 + q_0 q_1^2 + q_0 q_1^2 + q_0^4 q_1^2 + \dots) (1 + 2 q_0 q_1 + 5 q_0^2 q_1^2 + \dots) \end{aligned} \quad (3.18)$$

or

$$\begin{aligned} Z_{\mathbb{C}^2/\mathbb{Z}_3} &= \bullet_0 + \boxed{0} + \boxed{\begin{array}{c} 2 \\ 0 \end{array}} + \boxed{\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}} + \boxed{\begin{array}{c} 1 \\ 2 \\ 1 \\ 0 \end{array}} + \boxed{\begin{array}{cc} 2 & 1 \\ 0 & 1 \end{array}} + \boxed{\begin{array}{ccc} 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}} + \boxed{\begin{array}{c} 0 \\ 2 \\ 1 \\ 1 \\ 0 \end{array}} + \boxed{\begin{array}{cc} 2 & 1 \\ 0 & 1 \end{array}} + \boxed{\begin{array}{ccc} 2 & 0 & 1 \\ 0 & 1 & 1 \end{array}} + \boxed{\begin{array}{ccc} 2 & 1 & 2 \\ 0 & 1 & 2 \end{array}} + \boxed{\begin{array}{cccc} 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}} + \dots \quad (3.19) \\ &= \left(\bullet_0 + \boxed{0} + \boxed{\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}} + \boxed{\begin{array}{c} 2 \\ 0 \end{array}} + \boxed{\begin{array}{ccc} 2 & 1 & 2 \\ 0 & 1 & 2 \end{array}} + \dots \right) \left(\bullet_0 + \boxed{\begin{array}{c} 1 \\ 2 \\ 0 \end{array}} + \boxed{\begin{array}{cc} 2 & 1 \\ 0 & 1 \end{array}} + \boxed{\begin{array}{ccc} 2 & 1 & 2 \\ 0 & 1 & 2 \end{array}} + \dots \right) \\ &= (1 + q_0 + q_0 q_1 + q_0 q_2 + q_0 q_1 q_2^2 + q_0 q_1^2 q_2 + q_0 q_1^2 q_2^2 + \dots) (1 + 3 q_0 q_1 q_2 + \dots) \end{aligned}$$

and so on. The numbers in the boxes refer to the \mathbb{Z}_p representation under which the corresponding box (or D(-1)-instanton) transforms. A term $q_0^{k_0} q_1^{k_1} \dots$ represents a harmonic form in the moduli space component with k_a of a -type. Here we take $r = 0$ i.e. $N_a = \delta_{a,0}$. The remaining choices $r = 1, 2, \dots, p-1$ can be found from this by performing a cyclic permutations of the numbers in the starting boxes in Y and Y_{frac} . Notice that a regular diagram starts always with a box “0” according to its definition (3.16) and its k_a ’s are invariant under these cyclic permutations. As shown in [22], the pair $(Y_{\text{frac}}, Y_{\text{reg}})$ can be alternatively specified by the p sets (u_a, Y_a) with Young tableaux Y_a and integers

u_a satisfying $\sum_{a=0}^{p-1} u_a = N$. The regular and fractional part of a diagram Y can be extracted following a combinatoric algorithm developed in [22]. This algorithm is reviewed in appendix A.

$U(N)$ instantons are built by tensoring N copies of the $U(1)$ -instanton bundles. Now we start from the N tableaux set $\{(Y_\alpha, r_\alpha)\}$ in \mathbb{C}^2 , compute the character and restrict to \mathbb{Z}_p -invariant monomials. The tableaux (Y_α, r_α) can be again decomposed into its regular and fractional part $(Y_\alpha^{\text{reg}}|Y_\alpha^{\text{frac}}, r_\alpha)$ with $Y_\alpha^{\text{reg}}, Y_\alpha^{\text{frac}}$ tableaux of the regular and fractional type respectively and r_α are integers mod p . Alternatively the same information can be encoded in the pN set $(Y_{a\alpha}^*, u_{a\alpha})$ (see appendix A for details).

We say that a set Y_α is regular or fractional if all of the Y_α 's are of the same type. We remark that according to this definition, unlike the $U(1)$ case, the moduli space of a fractional instanton has not necessarily dimension zero since even for fractional tableaux Γ -invariant terms can appear in (3.6) from terms with $Y_{\text{frac},\alpha} \neq Y_{\text{frac},\beta}$. However these extra moduli are not associated to positions in M_4 (open string starting and ending on the same D(-1)-brane) and therefore fractional instantons are stuck at the singularity as expected.

Using (3.11), the instanton partition function (3.13) can be written as

$$\mathcal{Z}(q) = \sum_Y q^{k_0 + \frac{1}{2}C^{aa}N_a} e^{-z^a u_a} \tag{3.20}$$

It is important to notice that the Chern characters can be written as a sum over $U(1)$ contributions from Y_α

$$\begin{aligned} \text{ch}_1(\mathcal{E}_Y) &= \sum_\alpha \text{ch}_1(\mathcal{E}_{Y_\alpha}) = \sum_{\alpha,a} u_{a\alpha} \text{ch}_1(\mathcal{T}^a) \\ k &= k_0 + \frac{1}{2}C^{aa}N_a = \sum_\alpha (k_{0,\alpha} + \frac{1}{2}C^{r_\alpha r_\alpha}) \end{aligned} \tag{3.21}$$

with $k_{a\alpha}$ the number of instantons in Y_α transforming in representation R_a and

$$u_{a\alpha} = \delta_{r_\alpha,a} + k_{a+1,\alpha} + k_{a-1,\alpha} - 2k_{a,\alpha} \tag{3.22}$$

Notice that $u_{a\alpha}$ satisfy the constraint

$$\sum_a u_{a\alpha} = 1 \tag{3.23}$$

i.e. $u_{a\alpha}$ is characterized by $(p-1)N$ independent components $u_{a>0,\alpha}$. According to the factorization algorithm explained before the instanton fixed points can be completely characterized by the N-sets $\{Y_{a\alpha}^*, u_{a>0,\alpha}\}$ with $Y_{a\alpha}^*$, Np Young tableaux and a point in $u_{a\alpha} \in \mathbb{Z}^{N(p-1)}$.

The decomposition (3.21) implies that the $U(N)$ partition function factorizes into

$$\mathcal{Z}_{U(N)} = \mathcal{Z}_{U(1)}^N = (\mathcal{Z}_{U(1),\text{reg}} \mathcal{Z}_{U(1),\text{frac}})^N \tag{3.24}$$

This is not surprising since the Euler number of the instanton moduli space cannot depend on continuous deformations and therefore it can be computed in the completely broken $U(1)^N$ phase when D3-branes are far away from each other.

In the following we analyze the two type of instantons separately.

3.2 Regular Instantons

The moduli space of regular instantons can be described as follows. First consider the character (2.6) computed over the ring of holomorphic polynomials $\mathbb{C}[z_1, z_2]$ on \mathbb{C}^2 .

$$\chi_\epsilon(\mathbb{C}[z_1, z_2]) = \frac{1}{(1 - T_1)(1 - T_2)} = 1 + T_1 + T_2 + T_1^2 + T_2^2 + T_1 T_2 + \dots \quad (3.25)$$

Now we project the character onto those polynomials in \mathbb{C}^2 that are invariant under \mathbb{Z}_p . The result can be rewritten in the form

$$\begin{aligned} \chi_\epsilon(\mathbb{C}^\Gamma[z_1, z_2]) &= \frac{1}{p} \sum_{a=0}^{p-1} \frac{1}{(1 - \omega^a T_1)(1 - \omega^{-a} T_2)} \quad \omega = e^{\frac{2\pi i}{p}} \\ &= \sum_{a=0}^{p-1} \frac{1}{(1 - T_1^{p-a} T_2^{-a})(1 - T_1^{1-p+a} T_2^{1+a})} \end{aligned} \quad (3.26)$$

This implies that the space $\mathbb{C}^2/\mathbb{Z}_p$ can be thought of as p copies of \mathbb{C}^2 with coordinates (z_{1a}, z_{2a}) in each chart transforming as $(z_1^{p-a} z_2^{-a}, z_1^{1-p+a} z_2^{1+a})$. The character of the instanton moduli space can therefore be written as a sum over p copies of the \mathbb{C}^2 character (2.9)

$$\chi_\epsilon(\mathcal{M}_{Y_{\text{reg}}}) = \sum_{a=0}^{p-1} \sum_{\alpha, \beta}^N \mathcal{N}_{\alpha, \beta}^{Y_a^*}(T_1^{p-a} T_2^{-a}, T_1^{1-p+a} T_2^{1+a}) \quad (3.27)$$

This implies in particular that regular instantons are specified by p N Young tableaux $Y_a^* = \{Y_{a\alpha}^*\}$. The Y_a^* follows from Y_{reg} via the factorization algorithm explained in the last section. The instanton partition function computed using either (3.27) or (3.6) leads to the same results but expression (3.27) can be easily resummed to all instanton orders even in the general $\mathbb{C}^2/\Gamma_{p,q}$ case. In fact counting fixed points in the regular instanton moduli space character (3.27) boils down to count the number of partitions, $Y_{a\alpha}^*$, of the integers $k_{a\alpha}$. The precise relation between the two descriptions will be explained in appendix A.

Let us now compute the regular instanton partition function. First notice that $k_0 = k_1 = \dots = k_{p-1} = k$ implies $u_a = 0$, i.e. regular instantons have zero first Chern character and instanton number

$$k(\mathcal{M}_{\text{reg}}) = \sum_{a, \alpha} |Y_{a\alpha}^*| \quad (3.28)$$

where by $|Y_{a\alpha}^*|$, we denote the number of boxes in $Y_{a\alpha}^*$. The partition function is then given by the Np power of the number of Young tableaux, i.e. the number of partitions of k

$$Z_{\text{reg}} = \frac{1}{\hat{\eta}(q)^{Np}} \quad \hat{\eta}(q) \equiv \prod_{n=1}^{\infty} (1 - q^n) \quad (3.29)$$

Actually, we can do better and consider the cohomology of these spaces. For simplicity we take $N = 1$. As we explained before the Betti numbers $b_{2n_-} = \dim H^{2n_-}(\mathcal{M}_{\text{reg}})$ are given in terms of the number n_- of negative eigenvalues in (3.27) with the conditions (3.15). A simple inspection of (2.9) shows that negative eigenvalues can come only from the term

$$T_1^{-(p-a)h(s)-(v(s)+1)(p-a-1)} T_2^{ah(s)+(v(s)+1)(1+a)} \quad (3.30)$$

The eigenvalue happens to be negative whenever any of the following conditions is satisfied

- $a \neq p - 1$
- $a = p - 1$ and $h(s) > 0$

A row of length m in the Y_α tableau contributes then to the Poincaré Polynomial $q^m t^{2Nm}$ when $a \neq p - 1$ and $q^m t^{2N(m-1)}$ when $a = p - 1$. The generating function for the Poincaré polynomial can then be written as

$$P(t|\mathcal{M}_{\text{reg}}^{U(1)}) = \prod_{m=1}^{\infty} \frac{1}{(1 - q^m t^{2m})^{p-1} (1 - q^m t^{2m-2})} = \sum_{n=0}^{\infty} q^n P(t|(\mathbb{C}^2/\mathbb{Z}_p)^n/S_n)$$

in agreement with the fact that the moduli space of regular $U(1)$ instantons on $\mathbb{C}^2/\mathbb{Z}_p$ is given by the Hilbert scheme of points on $\mathbb{C}^2/\mathbb{Z}_p$ with $b_0 = 1, b_2 = p - 1$ (see [25] for the Poincaré polynomial symmetric product formula).

3.3 Fractional Instantons

Now we consider fractional instantons. We start with the $U(1)$ case. Fractional instantons correspond to Young tableaux containing no \mathbb{Z}_p -invariant box i.e. those tableau with no box $s = (i, j)$ with hook length satisfying $\ell(s) = \nu_j + \tilde{\nu}_i - i - j + 1 = 0 \pmod{p}$. Alternatively fractional instantons are defined by the condition

$$\dim_{\mathbb{C}} \mathcal{M}_Y^{U(1)} = 0 \quad \Rightarrow \quad k_r = \frac{1}{2} \widehat{C}_{ab} k_a k_b \quad (3.31)$$

This condition can be used to rewrite the instanton number k for fractional instantons entirely in terms of their first Chern classes u_a . To see this we first relate the dimension of the instanton moduli space to the square of its first Chern class ³

$$\begin{aligned} C^{ab} u_a u_b &= C^{ab} N_a N_b + C^{ab} \widehat{C}_{bc} \widehat{C}_{bd} k_c k_d - 2C^{ab} \widehat{C}_{bc} k_c N_d \\ &= C^{ab} N_a N_b + \widehat{C}_{ab} k_a k_b - 2k_a N_a + 2k_0 N \\ &= C^{ab} N_a N_b + 2k_0 N - \dim_{\mathbb{C}} \mathcal{M}_Y \end{aligned} \quad (3.32)$$

Then specializing to $U(1)$ fractional instantons, i.e. $N = 1$ and $\dim_{\mathbb{C}} \mathcal{M}_Y = 0$, one finds

$$k = k_0 + C^{aa} N_a = \frac{1}{2} C^{ab} u_a u_b \quad (3.33)$$

We recall that u_a spans \mathbb{Z}^{p-1} .

It is instructive to work it out some explicit examples. For $p = 2$ one finds

$$\begin{aligned} Z_{\text{frac}, N=(1,0)}^{\mathbb{Z}_2} &= \bullet_0 + \boxed{0} + \boxed{\begin{array}{c} 1 \\ 01 \end{array}} + \boxed{\begin{array}{c} 0 \\ 10 \\ 010 \end{array}} + \boxed{\begin{array}{c} 1 \\ 01 \\ 101 \\ 0101 \end{array}} + \dots \\ Z_{\text{frac}, N=(0,1)}^{\mathbb{Z}_2} &= \bullet_1 + \boxed{1} + \boxed{\begin{array}{c} 0 \\ 10 \end{array}} + \boxed{\begin{array}{c} 1 \\ 01 \\ 101 \end{array}} + \boxed{\begin{array}{c} 0 \\ 10 \\ 010 \\ 1010 \end{array}} + \dots \end{aligned} \quad (3.34)$$

Evaluating u_1

$$u_1 = 2(k_0 - k_1) + N_1 \quad (3.35)$$

³Here we used the identities $C^{ab} \widehat{C}_{a0} = C^{ab} \widehat{C}_{a0} \widehat{C}_{b0} = 1$.

for the diagrams in (3.34) one finds

$$\begin{aligned} r = 0 & \quad u_1 = 0, 2, -2, 4, -4, \dots \\ r = 1 & \quad u_1 = 1, -1, 3, -3, 5, \dots \end{aligned} \tag{3.36}$$

i.e. u_1 spans \mathbb{Z} in agreement with our general claim. The fractional instanton partition function can then be written as

$$\mathcal{Z}_{\text{frac}}^{\mathbb{Z}_2} = \sum_{u \in \mathbb{Z}} q^{\frac{1}{4}u^2} e^{-z^a u_a} \tag{3.37}$$

For $p = 3, r = 0$ one finds

$$\mathcal{Z}_{\text{frac}, N=(1,0,0)}^{\mathbb{Z}_3} = \bullet_0 + \boxed{0} + \left(\boxed{01} + \boxed{2} \right) + \left(\boxed{2} \boxed{012} + \boxed{2} \boxed{01} \right) + \boxed{2} \boxed{0112} + \dots \tag{3.38}$$

The results for $r = 1, 2$ follows from (3.38) by cyclic permutations of the labels. One can easily check that the resulting spectrum of $u_a = N_a + k_{a+1} + k_{a-1} - 2k_a$'s spans \mathbb{Z}^2 .

Using (3.33) the instanton partition function can then be written in general as

$$\mathcal{Z}_{\text{frac}}(q) = \sum_{u \in \mathbb{Z}^{p-1}} q^{\frac{1}{2}C^{ab}u_a u_b} e^{-z^a u_a} \tag{3.39}$$

The result for $U(N)$ is then given by the N power of (3.39)

$$\mathcal{Z}_{U(N), \text{frac}} = \sum_{\vec{u}_a \in \mathbb{Z}^{N(p-1)}} q^{\frac{1}{2}C^{ab}\vec{u}_a \cdot \vec{u}_b} e^{-z^a u_a} \tag{3.40}$$

with $\vec{u}_a \cdot \vec{u}_b \equiv \sum_{\alpha} u_{a\alpha} u_{b\alpha}$ and $u_a = \sum_{\alpha} u_{a\alpha}$. This result has been anticipated in [22] (see also [26] for the $U(1)$ case).

4. Instantons on toric varieties

Here we consider instantons on more general toric varieties. We start from the singular case. The most general toric singularity in four dimensions can be written as the quotient $\mathbb{C}^2/\Gamma_{p,q}$, with $\Gamma_{p,q}$ a \mathbb{Z}_p action specified by the two coprime integers $p, q < p$ with generator

$$\Gamma_{p,q} : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \rightarrow \begin{pmatrix} e^{2\pi i/p} & 0 \\ 0 & e^{2\pi i q/p} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{4.1}$$

and $z_{1,2}$ the complex coordinates on \mathbb{C}^2 . The two extreme cases $q = p - 1$ and $q = 1$ correspond to the familiar A_{p-1} and blown down $\mathcal{O}_{\mathbb{P}^1}(-p)$ surfaces respectively.

For generic (p, q) a smooth manifold is obtained by blowing up points to exceptional surfaces, C_i , whose intersection numbers are given expanding the fraction p/q as

$$\frac{p}{q} = e_1 - \frac{1}{e_2 - \frac{1}{\dots - \frac{1}{e_n}}} \tag{4.2}$$

in terms of the integers e_i . To each $i = 1, 2, \dots, n$ one associates the two-cycle C_i with self-intersection e_i . The intersection matrix is then given by [27]

$$C = \begin{pmatrix} e_1 & -1 & 0 & \dots & \dots & 0 \\ -1 & e_2 & -1 & 0 & \dots & 0 \\ 0 & -1 & e_3 & -1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & -1 & e_n \end{pmatrix} \quad (4.3)$$

The two extreme cases $q = p - 1$ and $q = 1$ lead to $n = p - 1, e_i = 2$ and $n = 1, e_1 = p$ respectively justifying the identification with the A_{p-1} and the blown down $\mathcal{O}_{\mathbb{P}^1}(-p)$ surfaces. For a general (p, q) one finds a “necklace” of n two-spheres. This space can be covered by $n + 1$ charts each looking locally like \mathbb{C}^2 .

We will consider $U(N)$ instantons on these spaces. Unlike the A_{p-1} case a description of the ADHM instanton moduli space for a general $\Gamma_{p,q}$ -singularity with $q \neq p - 1$ is not available in the literature (see [28–30] for results in the $\mathcal{O}_{\mathbb{P}^1}(-p)$ case for $p = 1, 2$). Still regular and fractional instantons on $M = \mathbb{C}^2/\Gamma_{p,q}$ can be easily constructed. As before, the $U(N)$ instanton partition function can be written in terms of that of $U(1)$ and therefore we can restrict ourselves to the $U(1)$ case. Regular $U(1)$ instantons are instantons transforming in the regular representation of \mathbb{Z}_p . They can move freely on M in sets of p -images symmetrically distributed around the singularity. The moduli space of k regular instantons is then given by specifying k -points on M up to permutations i.e. the Hilbert scheme $M^{[k]}$. Their contribution to the partition function is given by

$$Z_{\text{reg}}^{U(1)} = \sum_k q^k \chi(M^{[k]}) \equiv \hat{\eta}(q)^{-\chi(M)} \quad \hat{\eta}(q) = \prod_{n=1}^{\infty} (1 - q^n) \quad (4.4)$$

Next we consider instantons carrying non-trivial first Chern-class along the exceptional two-cycles. A self-dual $U(1)$ gauge field strength can be written as

$$\frac{iF}{2\pi} = u_i \alpha^i \in H_2^+(M, \mathbb{Z}) \quad (4.5)$$

with $\alpha^i, i = 1, \dots, b_2^+$ a basis $H_2^+(M, \mathbb{Z})$ and u_i some integers ⁴. These gauge connections are self-dual by construction and corresponds to isolated points in the moduli space since they do not admit any continuous deformation. We call them ”fractional”.

Their contribution to the Yang Mill action with the insertion of the observables in (3.13) can be written as

$$\begin{aligned} S_{SYM} &= -\frac{i\tau}{4\pi} \int_M F \wedge F - \varphi_{2i} \int \frac{iF}{2\pi} \wedge \alpha^i \\ &= -\pi i \tau C^{ij} u_i u_j + z^i u_i \end{aligned} \quad (4.6)$$

with $\tau = \frac{4\pi i}{g_{\text{YM}}^2} + \frac{\theta}{2\pi}$ and $z^i = C^{ij} \varphi_{2j}$. The fractional instanton partition function $\sum_u e^{-S_{YM}}$ can then be written as

$$Z_{\text{frac}}^{U(1)} = \sum_{u_i \in \mathbb{Z}^{b_2^+(M)}} q^{\frac{1}{2} C^{ij} u_i u_j} e^{-z^i u_i} \quad (4.7)$$

⁴In the A_{p-1} case one can choose $\alpha^i = c_1(T^i)$ as the basis.

Collecting the regular and fractional instanton contributions one finds the partition function

$$Z = (Z_{\text{reg}}^{U(1)} Z_{\text{frac}}^{U(1)})^N = \frac{1}{\hat{\eta}(q)^{N\chi(M)}} \sum_{\vec{u}_i \in \mathbb{Z}^N b_2^+(M)} q^{\frac{1}{2} C^{ij} u_{i\alpha} u_{j\alpha}} e^{-z^i u_i} \quad (4.8)$$

where $u_i = \sum_{\alpha=1}^N u_{i\alpha}$, $\chi(M)$, $b_2^+(M)$ are the Euler number and the number of self-dual forms in M respectively and C^{ij} is the inverse of the intersection matrix given by (4.3). Specifying to the A_{p-1} case, $\chi = b_2^+ + 1 = p$ one finds the result (3.39).

In this paper we mainly focus on non-compact toric varieties but we should stress that our considerations extend naturally to the compact case. Indeed, in general a toric variety can be covered by $\chi(M)$ charts each looking locally as \mathbb{R}^4 and one can think of regular $U(N)$ instantons on M as $\chi(M)$ copies of $U(N)$ instantons on \mathbb{R}^4 . Equivalently they can be thought of as N copies of $U(1)$ instantons described by the Hilbert schemes $M^{[k]}$. Their contribution to the partition formula is $\hat{\eta}^{-\chi N}$. Fractional instantons can be built out of integer valued linear combinations $F = u_i \alpha^i$ of the self-dual forms α^i forming a basis $H_2^+(M, \mathbb{Z})$ and contribute to the lattice sum in (4.8).

For a general toric variety the ADHM construction of self-dual connections is not known. In general we cannot ensure that any self-dual connections can be entirely built out of regular and fractional instantons of the type describe here and therefore extra contributions to (4.8) cannot be excluded. Nonetheless the tests we will perform against 2d SYM computations and supergravity D0-D2-D4 black entropy formulae suggest that (4.8) holds (even for compact manifolds) at least in the limit where instanton charges are taken to be large.

4.1 Instantons on $O_{\mathbb{P}_1}(-p)$

As an illustration consider the case $q = 1$: the blown down $O_{\mathbb{P}_1}(-p)$ surface.

Regular instantons

The ring of invariant polynomials in \mathbb{C}^2 under $\Gamma = \Gamma_{p,1}$ can be written as

$$\begin{aligned} \chi_\epsilon(\mathbb{C}^\Gamma[z_1, z_2]) &= \frac{1}{p} \sum_{a=0}^{p-1} \frac{1}{(1 - \omega^a T_1)(1 - \omega^a T_2)} \quad \omega = e^{\frac{2\pi i}{p}} \\ &= \frac{1}{(1 - T_1^p)(1 - \frac{T_2}{T_1})} + \frac{1}{(1 - T_2^p)(1 - \frac{T_1}{T_2})} \end{aligned} \quad (4.9)$$

This implies that the space \mathbb{C}^2/Γ can be thought of as two copies of \mathbb{C}^2 with coordinates $(z_1^p, \frac{z_2}{z_1})$ and $(\frac{z_1}{z_2}, z_2^p)$ in the two patches. The instanton moduli space character can therefore be written as a sum of two \mathbb{C}^2 characters

$$\chi_\epsilon(\mathcal{M}_{Y_{\text{reg}}}) = \sum_{\alpha, \beta}^N \left[\mathcal{N}_{\alpha, \beta}^{Y_1} \left(T_1^p, \frac{T_2}{T_1} \right) + \mathcal{N}_{\alpha, \beta}^{Y_2} \left(\frac{T_1}{T_2}, T_2^p \right) \right] \quad (4.10)$$

specified by the $2N$ Young tableaux $\{Y_{1\alpha}, Y_{2\alpha}\}$. The partition function is then given by the $2N$ power of the number of partitions of k

$$Z_{\text{reg}} = \frac{1}{\hat{\eta}(q)^{2N}} \quad (4.11)$$

The Poincaré polynomial can be computed as before counting the number of negative eigenvalues in (4.10). For $N = 1$ there is a negative eigenvalue whenever any of the following conditions are satisfied

- Y_1 : for each box
- Y_2 : $h(s) > 0$

The Poincaré generating polynomial can then be written as

$$P(t|\mathcal{M}_{\text{reg}}^{U(1)}) = \prod_{m=1}^{\infty} \frac{1}{(1 - q^m t^{2m})(1 - q^m t^{2m-2})} = \sum_{n=0}^{\infty} q^n P(t|(\mathbb{C}^2/\Gamma)^n/S_n)$$

in agreement with the fact that the moduli space of regular $U(1)$ instantons on $\mathbb{C}^2/\Gamma_{p,1}$ is given by the Hilbert scheme of points on $\mathbb{C}^2/\Gamma_{p,1}$ with $b_0 = b_2 = 1$.

Fractional instantons

Fractional instanton connections can be constructed as before in terms of integer-valued combinations of the self-dual two forms on \mathbb{C}^2/Γ . In the $O_{\mathbb{P}^1}(-p)$ there is a single two-form $c_1(\mathcal{T})$ with self-intersection

$$\int c_1(\mathcal{T}) \wedge c_1(\mathcal{T}) = -\frac{1}{p} \tag{4.12}$$

The fractional self dual connection can then be written as

$$\frac{iF}{2\pi} = \text{diag}(u_1, u_2, \dots, u_N) c_1(\mathcal{T}) \tag{4.13}$$

with Yang-Mill action

$$\begin{aligned} S_{SYM} &= -\frac{i\tau}{4\pi} \int_M \text{tr} F \wedge F + \varphi_2 \int \text{tr} \frac{iF}{2\pi} \wedge c_1(\mathcal{T}) \\ &= -\frac{\pi i\tau}{p} \vec{u} \cdot \vec{u} + \frac{1}{p} u \varphi_2 \end{aligned} \tag{4.14}$$

and $u = \sum_{\alpha} u_{\alpha}$. The partition function can then be written as

$$\mathcal{Z}_{\text{frac}} = \sum_{u_{\alpha} \in \mathbb{Z}^N} q^{\frac{1}{2p} \vec{u} \cdot \vec{u}} e^{-zu} \quad z = \frac{\varphi_2}{p} \tag{4.15}$$

4.2 Instantons on $\mathbb{C}^2/\Gamma_{5,2}$

Finally we consider the singularity $\mathbb{C}^2/\Gamma_{5,2}$. This singularity does not belong neither to the A_{p-1} nor to the $O_{\mathbb{P}^1}(-p)$ series and illustrates the general case.

Regular instantons

The ring of invariant polynomials in \mathbb{C}^2 under $\Gamma = \Gamma_{5,2}$ can be written as

$$\begin{aligned} \chi_{\epsilon}(\mathbb{C}^{\Gamma}[z_1, z_2]) &= \frac{1}{p} \sum_{a=0}^4 \frac{1}{(1 - \omega^a T_1)(1 - \omega^{2a} T_2)} \quad \omega = e^{\frac{2\pi i}{5}} \\ &= \frac{1 + T_1 T_2^2 + T_1^2 T_2^4 + T_1^3 T_2 + T_1^4 T_2^3}{(1 - T_1^5)(1 - T_2^5)} \\ &= \frac{1}{(1 - T_1^5)(1 - \frac{T_2}{T_1})} + \frac{1}{(1 - \frac{T_1^2}{T_2})(1 - \frac{T_2^3}{T_1})} + \frac{1}{(1 - \frac{T_1}{T_2^3})(1 - T_2^5)} \end{aligned} \tag{4.16}$$

This implies that the space $\mathbb{C}^2/\Gamma_{5,2}$ can be thought of as three copies of \mathbb{C}^2 with coordinates $(z_1^5, \frac{z_2}{z_1^2})$, $(\frac{z_1^2}{z_2}, \frac{z_2^3}{z_1})$ and $(\frac{z_1}{z_2^3}, z_2^5)$ on the three patches.

The regular instanton moduli space character can therefore be written as a sum of three \mathbb{C}^2 characters

$$\chi_\epsilon(\mathcal{M}_{Y_{\text{reg}}}) = \sum_{\alpha,\beta}^N \left[\mathcal{N}_{\alpha,\beta}^{Y_1} \left(T_1^5, \frac{T_2}{T_1} \right) + \mathcal{N}_{\alpha,\beta}^{Y_1} \left(\frac{T_1^2}{T_2}, \frac{T_2^3}{T_1} \right) + \mathcal{N}_{\alpha,\beta}^{Y_2} \left(\frac{T_1}{T_2^3}, T_2^5 \right) \right] \quad (4.17)$$

specified by the $3N$ Young tableaux $\{Y_{1\alpha}, Y_{2\alpha}, Y_{3\alpha}\}$. The partition function is then given by the $3N$ power of the number of partitions of the integer k

$$Z_{\text{reg}} = \frac{1}{\hat{\eta}(q)^{3N}} \quad (4.18)$$

The Poincaré polynomial can be computed as before counting the number of negative eigenvalues in (4.17). For $N = 1$ there is a negative eigenvalue whenever any of the following conditions are satisfied

- Y_1, Y_2 : for each box
- Y_3 : $h(s) > 0$

The generating function of the Poincaré polynomial can then be written as

$$P(t|\mathcal{M}_{\text{reg}}^{U(1)}) = \prod_{m=1}^{\infty} \frac{1}{(1 - q^m t^{2m})^2 (1 - q^m t^{2m-2})} = \sum_{n=0}^{\infty} q^n P(t|(\mathbb{C}^2/\Gamma_{5,2})^n/S_n)$$

in agreement with the fact that the moduli space of regular $U(1)$ instantons on $\mathbb{C}^2/\Gamma_{5,2}$ is given by the Hilbert scheme of points on $\mathbb{C}^2/\Gamma_{5,2}$ with $b_0 = 1, b_2 = 2$.

Fractional instantons

Fractional instanton connections can be constructed as before in terms of integer-valued combinations of the self-dual two forms $c_1(\mathcal{T}^i)$, $i = 1, 2$, on $\mathbb{C}^2/\Gamma_{5,2}$ with self-intersection

$$\int c_1(\mathcal{T}^i) \wedge c_1(\mathcal{T}^j) = -C^{ij} \quad (4.19)$$

with C^{ij} the inverse of (4.3)

$$C_{ij} = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \quad C^{ij} = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \quad (4.20)$$

The fractional self dual connection can then be written as

$$\frac{iF}{2\pi} = \text{diag}(u_{i1}, u_{i2}, \dots, u_{iN}) c_1(\mathcal{T}^i) \quad (4.21)$$

with Yang-Mill action

$$\begin{aligned} S_{SYM} &= -\frac{i\tau}{4\pi} \int_M \text{tr} F \wedge F + \varphi_{2i} \int \text{tr} \frac{iF}{2\pi} \wedge c_1(\mathcal{T}^i) \\ &= -\pi i \tau C^{ij} \vec{u}_i \cdot \vec{u}_j + C^{ij} u_i \varphi_{2j} \end{aligned} \quad (4.22)$$

and $u_i = \sum_{\alpha} u_{i\alpha}$. The partition function can then be written as

$$\mathcal{Z}_{\text{frac}} = \sum_{u_{i\alpha} \in \mathbb{Z}^{2N}} q^{\frac{1}{2} C^{ij} \vec{u}_i \cdot \vec{u}_j} e^{-z^i u_i} \quad z^i = C^{ij} \varphi_{2j} \quad (4.23)$$

5. Black hole partition functions

In this section we derive a microscopic formula for the partition function of a black hole made out of D4-D2-D0 bound states wrapping a four cycle inside a CY. We will restrict ourselves to the case where both the cycle and the CY are compact. The lift of this brane system to M-theory is well known and a microscopic derivation of the corresponding black hole entropy based on a two-dimensional (4,0) SCFT has been derived in [33] (see also [34–38] for related (macro)microscopic countings). The aim of this section is to test our instanton partition function formula against supergravity. We refer the readers to [33] for details on the geometrical tools we will use here. We consider a single D4-brane wrapping a very ample divisor P inside a CY. The conjugacy class $[P] \in H^2(CY, \mathbb{Z})$ can be expanded as $[P] = p^A \alpha_A$ with α_A a basis in $H^2(CY, \mathbb{Z})$.

According to [1] the black hole partition function is defined as

$$Z_{BH} = \sum_{Q_0, Q_A} \Omega(Q_0, Q_A, p^A) e^{-Q_0 \varphi_0 - Q_A \varphi^A} \quad (5.1)$$

with $\Omega(Q_0, Q_A, p^A)$ the multiplicity of a bound state of Q_0 D0-branes, Q_A D2-branes and a D4 brane wrapping $P = p^A \Sigma_A$. φ_0, φ^A are the D0, D2 chemical potentials. D0, D2 branes can be thought of as instantons and fluxes respectively in the worldvolume theory of the D4-brane

$$Q_0 = k = \frac{1}{8\pi^2} \int_M \text{tr } F \wedge F \quad Q_A = C_{AB} u^B = \frac{1}{2\pi} \int \text{tr } F \wedge \alpha_A \quad (5.2)$$

Self-duality implies that $Q_{A > b_2^+} = 0$.

The black hole partition function can be then read from the instanton partition function formula (4.8)

$$\begin{aligned} Z_{BH} &= \frac{1}{\hat{\eta}(q)^{\chi(P)}} \sum_{u^A \in \mathbb{Z}^{b_2^+(P)}} q^{\frac{1}{2} C_{AB} u^A u^B} e^{-\varphi_A u^A} \\ &= \frac{1}{\hat{\eta}(e^{-\varphi_0})^{\chi(P)}} \sum_{Q_A \in \mathbb{Z}^{b_2^+(P)}} e^{-\frac{1}{12} D^{AB} Q_A Q_B \varphi_0 - \varphi^A Q_A} \\ &= \sum_{Q_0, Q_A} \Omega(Q_0, Q_A, p^A) q^{Q_0} e^{-\varphi^A Q_A} \end{aligned} \quad (5.3)$$

with

$$\begin{aligned} \chi(P) &= \int_P c_2(P) = D_{ABC} p^A p^B p^C + c_{2A} p^A \\ b_2^+(P) &= 2 D_{ABC} p^A p^B p^C + \frac{1}{6} c_{2A} p^A \\ D_{ABC} &\equiv \frac{1}{6} \int_{CY} \alpha_A \wedge \alpha_B \wedge \alpha_C \quad c_{2A} \equiv \int_{CY} \alpha_A \wedge c_2(CY) \\ C_{AB} &= - \int_P \alpha_A \wedge \alpha_B = -6 D_{AB} \quad D_{AB} \equiv D_{ABC} p^C \\ q &= e^{-\varphi_0} \quad e^{-z_A} = e^{-C_{AB} \varphi^B} \end{aligned} \quad (5.4)$$

and C^{AB}, D^{AB} the inverse of C_{AB}, D_{AB} respectively.

Notice that (5.3) is the partition function of $\chi(P)$ free bosons (b_2^+ of them living in the lattice $H^2(P, \mathbb{Z})$) in two-dimensions. The black hole entropy follows from the Cardy formula

$$\begin{aligned}
 S_{BH} &\approx \ln \Omega(Q_0, Q_A, p^A) \approx 2\pi \sqrt{\frac{1}{6} \chi(P) Q_{0,\text{reg}}} \\
 &= 2\pi \sqrt{(D_{ABC} p^A p^B p^C + \frac{1}{6} c_{2A} p^A)(Q_0 + \frac{1}{12} D^{AB} Q_A Q_B)} \quad (5.5)
 \end{aligned}$$

with $Q_{0,\text{reg}} = Q_0 + \frac{1}{12} D^{AB} Q_A Q_B$ the number of regular instantons coming from the expansion of $\hat{\eta}^{-\chi}$ in (5.3). (5.5) agrees with the micro/macrosopic M5-brane/supergravity results found in [33].

6. Summary and conclusions

In this paper we have studied instantons on toric varieties. Starting from the ADHM construction of instanton on A_{p-1} we derived the instanton partition function formula

$$Z = (Z_{\text{reg}}^{U(1)} Z_{\text{frac}}^{U(1)})^N = \frac{1}{\hat{\eta}(q)^{N \chi(M)}} \sum_{\vec{u}_i \in \mathbb{Z}^N b_2^+(M)} q^{\frac{1}{2} C^{ij} u_{i\alpha} u_{j\alpha}} e^{-z^i u_i} \quad (6.1)$$

where $u_i = \sum_{\alpha=1}^N u_{i\alpha}$, $\chi(M) = p$, $b_2^+(M) = p - 1$ are the Euler number and the number of self-dual forms in M respectively and C^{ij} is the inverse of the intersection matrix i.e. minus the Cartan matrix of A_{p-1} . That the Euler number of the moduli space of $U(N)$ instantons can be related to that of $U(1)$ instantons follows from the fact that that after localization $U(N)$ is broken to $U(1)^N$. There are two very special classes of $U(1)$ instantons: *regular* and *fractional*. Regular instantons are instantons transforming in the regular representation of \mathbb{Z}_p . They can move freely on M in sets of p -images symmetrically distributed around the singularity. The moduli space can then be thought of as the Hilbert scheme of k points in M contributing $\eta^{-\chi}$ to the partition function. Alternatively, covering M with $\chi(M)$ charts, $U(N)$ instantons on M can be thought as $\chi(M)$ copies of $U(N)$ instantons on \mathbb{R}^4 each contributing $\hat{\eta}^{-N}$.

Fractional instantons are instantons with no or incomplete images. They are stuck at the singularity since all open strings moduli starting and ending on the same D(-1)-brane are projected out by the orbifold. They carry non-trivial fluxes along the exceptional two cycles and correspond to self-dual connections $\frac{iF}{2\pi} = u_a \alpha^a$ where the α^a form a basis in $H^{2+}(M|\mathbb{Z})$. Their contribution to the partition function gives the lattice sum in (6.1).

For a general toric variety M , an ADHM construction is missing but still regular and fractional instantons can be constructed exactly in the same way as in the A_{p-1} case⁵.

⁵The fact that the instanton partition function factorizes into a product of regular and fractional instantons has been experimentally observed in [20] for instantons A_{p-1} -singularity and shown in [22]. For general (p, q) , there is no proof that this should be the case and we don't exclude that extra contributions can arise from new classes of instantons. The perfect agreement with 2d SYM computations and supergravity however suggests that this is not the case.

Here we work out in details the case of toric singularities of type $M = \mathbb{C}^2/\Gamma_{p,q}$ with $\Gamma_{p,q}$ a \mathbb{Z}_p action specified by the coprime integers (p, q) . The blown down $O_{\mathbb{P}^1}(-p)$ surface corresponds to the case $q=1$, $\chi(M) = b_2^+ + 1 = 2$. It is interesting to notice that the result (6.1) corresponds to the chiral partition function of a two-dimensional conformal field theory with central charge $N\chi$. This is not surprising from the M-theory perspective where the D4-D2-D0 quantum mechanics lift to a $(4, 0)$ two-dimensional SCFT living on the M5-brane worldvolume [33].

Comparing our results with previous computations based on 2d SYM [5, 6, 31] one finds that only the contributions of fractional instantons are captured by the 2d analysis. This is not surprising since the 2d SYM picture focalizes on the neighborhood of the singularity and can hardly trace regular instantons that can move freely far away from the singularity. For the $U(1)$ case, the fractional instanton lattice sum in (6.1) is in perfect agreement with the 2d SYM results in the case of A_{p-1} and $O_{\mathbb{P}^1}(-p)$ surfaces. More precisely taking $\chi(M) = p$ and $\chi(M) = 2$ in (6.1) one finds the Poisson resummation version of formula(6.7) in [6] and formula (27) of [31] respectively. Formula (6.1) extends these results to the general (p, q) -case and completes them with the inclusion of regular instanton contributions that are crucial to reproduce the right black hole entropy.

The comparison for $N > 1$ is trickier. The formulae in [6, 31] include perturbative corrections in the gauge coupling, g_{YM} , which cannot be interpreted as a topological number such as the Euler number of an instanton moduli space. These corrections come from the Chern-Simon contributions on the boundary of the non-compact space $\mathbb{C}^2/\Gamma_{p,q}$. Their significance has been very recently clarified in [32]. The two-dimensional SYM partition function has been written in a product form involving an instanton sum and a Chern-Simon contribution from the boundary Lens space in [31, 32]. Remarkably, discarding the semiclassical fluctuations in the Chern-Simon theory, the two results perfectly agree, see (3.23) of [32].

Finally we compute the partition function of D0-D2-D4 bound state wrapping a compact four cycle M with conjugacy class $[M] = p^A \alpha_A \in H^2(CY|\mathbb{Z})$ inside a CY. Applying the Cardy formula to estimate the growth of the bound state multiplicities in (6.1), at large Q_0 one finds

$$S_{BH} = \ln \chi(\mathcal{M}_{Q_0, Q_B}) \approx 2\pi \sqrt{(D_{ABCP^A} p^B p^C + \frac{1}{6} c_{2AP^A})(Q_0 + \frac{1}{12} D^{AB} Q_A Q_B)} \quad (6.2)$$

with $\chi(\mathcal{M}_{Q_0, Q_A})$ the moduli space of Q_0 instantons with first Chern class Q_A for an $\mathcal{N} = 4$ SYM living on the four-cycle M with conjugacy class $[M] = p^A \alpha_A$ inside the CY. Formula (6.2) perfectly agrees with the micro/macrosopic M5-brane/supergravity results found in [33].

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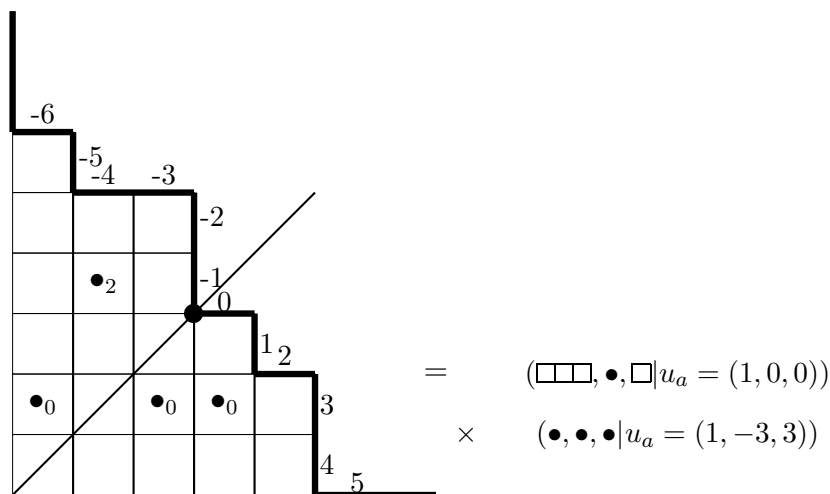


Figure 1: Factorization algorithm. The diagrams on the right hand side (Y_a^*, u_a) represents the regular (upper) and fractional part (bottom) of tableau Y in the left hand side. The embedding of the tableaux Y_a^* (in the r.h.s. of the figure) is given by the boxes marked by \bullet_a in Y .

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A. Factorization algorithm

The regular and fractional part of a diagram Y can be extracted following a combinatoric algorithm developed in [22]. First, one associates to the Young tableau Y a sequence $m_Y(n)$, $n \in \mathbb{Z}$ made of 0’s and 1’s. The sequence is constructed as follows: to each horizontal (vertical) segment in the Young tableau profile, we assign a 1 (0). The term $n = 0$ in the sequence is given by the segment to the right of the middle point given by the intersection between the profile and the main diagonal, see figure 1 where the middle point is marked by a bullet. n grows from left to right along the Young tableau profile from $-\infty$ to ∞ . Notice that a sequence built in this way satisfy $m_Y(-\infty) = 0$ and $m_Y(\infty) = 1$. Conversely a sequence satisfying these boundary conditions specifies uniquely a Young tableau. For example the Young tableau of figure 1, representing the partition $\nu^Y = (5, 5, 4, 3, 3, 1)$, leads to the sequence $m_Y(n)$ in the first line in table 1. Next we split the sequence $m_Y(n)$ into p subsequences $m_{Y_a^*}$ according to the *mod p* parity of n , i.e.

$$m_{Y_a^*}(n) = m_Y(np + a) \tag{A.1}$$



n	$-\infty$.	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	.	∞	Y_a^*	ν_a	u_a
m_Y	0	0	0	1	0	1	1	0	0	1	0	1	0	0	1	1	1	x	x	x
$m_{Y_0^*}$	0			1			1			1			0			1			-1	1
$m_{Y_1^*}$	0	0			0			0			0			0			1	•	2	-3
$m_{Y_2^*}$	0		0			1			0			1				1			-1	3

Table 1: The sequences m_Y or $m_{Y_a^*}$ specify uniquely the Young tableaux Y in figure 1 and its regular part $(Y_a^*, u_a = (1, 0, 0))$. ν_a or equivalently u_a describe the fractional part carrying the non-trivial first Chern class .

The resulting subsequences are displayed in table 1. Each subsequence $m_{Y_a^*}$ describes a new Young tableau profile that we denote by Y_a^* , see table 1. In addition the first Chern classes, u_a , can then be read from

$$u_a = \nu_{a+r-1} - \nu_{a+r} + \delta_{a,r} \tag{A.2}$$

with

$$\nu_a \equiv \# \{m_{Y_a^*}(n) = 0 | n \geq 0\} - \# \{m_{Y_a^*}(n) = 1 | n < 0\} \tag{A.3}$$

We remark that from this definition $\sum_a \nu_a = 0$ since it gives the difference between the number of horizontal segments to the left and the number of vertical segments on the right of the middle point. Therefore giving the $(p - 1)$ independent components u_a 's or the ν_a 's is equivalent.

It is interesting to remark that the subdiagrams Y_a^* can be thought of as embedded inside Y . To see this first let us note that to each invariant box contributing non trivially to the character (3.6) it is associated a hook starting and end whose on segments labelled by n_{up} and n_{right} satisfying $n_{up} = n_{right} = a(mod p)$ for some $a = 0, \dots, p-1$. Let us indicate each invariant box by a bullet with an index a (e.g. for the bullet with index 2 in figure 1 we have $n_{up} = -4 = 2(mod 3)$, $n_{right} = -1 = 2(mod 3)$). The bullets of index a precisely indicate the embedding of Y_a^* inside Y (see figure 1). In general the images of Y_a^* in Y overlap, but in the special case when u_a are large enough the diagrams Y_a^* become well separated. It is not hard to see that in such cases the characters (3.6) and (3.27) coincide. Though generally speaking the characters (3.6) and (3.27) are different they give rise to the same results when $\epsilon_1 = -\epsilon_2$ as well as to the same Poincare' polynomials

For the tableau of figure 1 the results of (A.2) and (A.3) are displayed in the last two columns of table 1. Conversely given the set $\{Y_a^*, u_a\}$ one can reconstruct the sequence $m_Y(n)$ and therefore the tableau Y , i.e. the algorithm gives a one-to-one correspondence between a Young tableau Y and the p sets (Y_a^*, u_a) with $\sum_a u_a = 1$.

The regular and fractional part of a diagram can be extracted from (Y_a^*, u_a) by the inverse algorithm via the identification

$$Y_{reg} \leftrightarrow (Y_a^*, u_a = \delta_{a,0}) \quad Y_{frac} \leftrightarrow (Y_a^* = \bullet, u_a) \tag{A.4}$$

i.e. from $(Y_a^*, u_a = \delta_{a,0})$ one reconstructs the subsequences $m_{Y_{a,reg}^*}(n)$ and from them the profile given by $m_{Y_{reg}}(n)$ and the tableau Y_{reg} . More precisely, since $u_a = \delta_{a,0}$ implies $\nu_a = 0$, $m_{Y_{a,reg}^*}(n)$ are found by translating the subsequences $m_{Y_a^*}(n)$ in table 1 in such a way that the number of 0's for $n \geq 0$ is the same as that of 1's for $n < 0$.

To extract Y_{frac} it is easier. Y_{frac} can be found by removing from Y all possible hooks of length a multiple of p . Indeed according to (3.10) u_a is invariant under the operation of removing a hook of length lp or $k_a \rightarrow k_a - l$. Therefore the fractional tableau obtained with this operation carries the same u_a of Y . For the diagram in fig 1 one finds

$$\text{Diagram} = \text{Diagram} \times \text{Diagram} \tag{A.5}$$

We have indicated by a bullet the invariant boxes contributing to the character. Y_{frac} is the diagram without bullets and is found by removing the length 9 and length 3 hooks containing the bullets in Y (the diagram in the left hand side). Notice that both Y and Y_{reg} has as many invariant boxes as boxes in Y_a^* .

B. $\mathbb{C}^2/\Gamma_{p,q}$ toric geometry

In this appendix we review the geometry of toric singularities $\mathbb{C}^2/\Gamma_{p,q}$. A complete treatment of this subject can be found in [39]. A toric variety of complex dimension d is specified by a cone in \mathbb{R}^d generated by a set of vectors \vec{v}_a (with integer coefficients)

$$\sigma \equiv \left\{ \sum_a r_a \vec{v}_a \mid r_a \in \mathbb{R}_+^d \right\} \tag{B.1}$$

Given σ , one introduces the dual cone σ^* , and the lattice reductions σ_N, σ_N^*

$$\begin{aligned} \sigma^* &\equiv \left\{ \vec{u} \in \mathbb{R}^d \mid \vec{u} \cdot \vec{v} \geq 0 \quad \forall \vec{v} \in \sigma \right\} \\ \sigma_N &\equiv \sigma \cap \mathbb{Z}^d \quad \sigma_N^* \equiv \sigma^* \cap \mathbb{Z}^d \end{aligned} \tag{B.2}$$

The lattices σ_N, σ_N^* encode the basic geometrical data of the toric variety. In particular, points on σ_N^* are in one-to-one correspondence with the set of holomorphic functions on the toric variety. More precisely, let $\{(a_{1m}, \dots, a_{dm})\}$ a basis for σ_N^* i.e. the minimal set of vectors in σ_N^* such that any point in σ_N^* can be written as $\sum_{m=1}^D c_m (a_{1m}, \dots, a_{dm})$ with $c_m \in \mathbb{Z}_+$. Then the ring of holomorphic functions on the toric variety can be written as

$$\mathbb{C}[\{x_m\}]/G = \mathbb{C}[\{w_1^{a_{1m}} w_2^{a_{2m}} \dots w_d^{a_{dm}}\}] \quad m = 1, \dots, D \tag{B.3}$$

The variables x_m are clearly not independent, they satisfy $D - d$ relations G that can be used to define the variety as an hypersurface on \mathbb{C}^D . Some basic properties are evident in σ_N . A variety is non-singular if and only if any point inside σ_N can be written as an integer-valued linear combination of $\{v_a\}$. Clearly the toric variety can be made regular by adding enough \vec{v}_a 's to the cone σ . A variety is compact if σ_N is isomorphic to \mathbb{Z}^d .

Let us illustrate these abstract notions in the case of $\mathbb{C}^2/\Gamma_{p,q}$ -singularities, (p, q) being coprime numbers with $q < p$. More precisely $\Gamma_{p,q}$ is a \mathbb{Z}_p action generated by

$$\Gamma_{p,q} : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \rightarrow \begin{pmatrix} e^{2\pi i q/p} & 0 \\ 0 & e^{2\pi i/p} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{B.4}$$

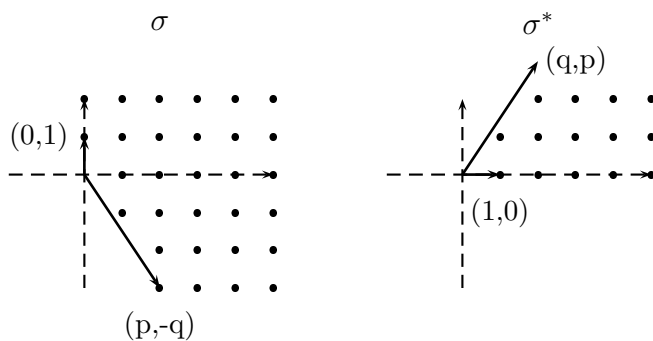


Figure 2: Toric diagram for the $\mathbb{C}^2/\Gamma_{p,q}$ singularity.

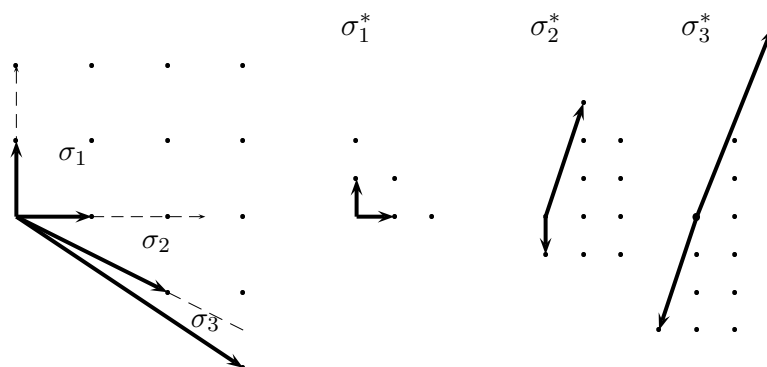


Figure 3: Resolution of $\mathbb{C}^2/\Gamma_{3,2}$.

with $z_{1,2}$ the complex coordinates on \mathbb{C}^2

The cone σ and its dual σ^* in this case are generated by the vectors \vec{v}_a, \vec{v}_a^* given by (see fig 2)

$$\begin{aligned} \vec{v}_0 &= (0, 1) & \vec{v}_1 &= (p, -q) \\ \vec{v}_0^* &= (1, 0) & \vec{v}_1^* &= (q, p) \end{aligned} \tag{B.5}$$

The variables w_1, w_2 are built out of invariant combinations of z_1, z_2

$$w_1 = z_1^p \quad w_2 = \frac{z_2}{z_1^q} \tag{B.6}$$

A_{p-1} -singularity

Take $q = p - 1$. Collecting the basic monomials in σ_N^* one finds

$$\mathbb{C}[x_1, x_2, x_3]/G = \mathbb{C}[w_1, w_1^{p-1} w_2^p, w_1 w_2] \tag{B.7}$$

with

$$G : \quad x_1 x_2 = x_3^p \tag{B.8}$$

This equation realizes the orbifold as a hypersurface on \mathbb{C}^3 .

The variety can be made regular adding vectors v_a ⁶

$$\sigma : \quad \{v_a = (a, 1 - a), \quad a = 0, \dots, p - 1\} \quad (\text{B.9})$$

The new cone become the union of p -cones σ_a defined by (see figure 3)

$$\begin{aligned} \sigma_a &= \{(a, 1 - a), (a + 1, -a)\} \quad a = 0, \dots, p \\ \sigma_a^* &= \{(1 - a, -a), (a, a + 1)\} \end{aligned} \quad (\text{B.10})$$

This corresponds to blow up $(p - 1)$ \mathbb{P}_1 's one for each extra v_a . The cone σ_N^* is made out of p cones $\sigma_{N,a}^*$ with polynomial ring

$$\sigma_N^* : \quad \bigoplus_a \mathbb{C}[w_1^{1-a} w_2^{-a}, w_1^a w_2^{a+1}] = \bigoplus_a \mathbb{C}[z_1^{p-a} z_2^{-a}, z_1^{a+1-p} z_2^{a+1}] \quad (\text{B.11})$$

with

$$w_1 = z_1^p \quad w_2 = \frac{z_2}{z_1^{p-1}} \quad (\text{B.12})$$

This is precisely the result we found for the A_{p-1} -polynomial ring in (3.26).

$O_{\mathbb{P}_1}(-p)$ -singularity

Take $q = 1$. The resolved variety $O_{\mathbb{P}_1}(-p)$ is described by the union of 2-cones σ_1 and σ_2 defined by

$$\begin{aligned} \sigma_1 &= \{(0, 1), (1, 0)\} & \sigma_2 &= \{(1, 0), (p, -1)\} \\ \sigma_1^* &= \{(1, 0), (0, 1)\} & \sigma_2^* &= \{(0, -1), (1, p)\} \end{aligned} \quad (\text{B.13})$$

This corresponds to blow up of a \mathbb{P}_1 's corresponding to the extra $v_2 = (0, 1)$. The cone σ_N^* is made out of 2 cones with polynomial ring

$$\mathbb{C}[w_1, w_2] \oplus \mathbb{C}[w_2, w_1 w_2^p] = \mathbb{C}[z_1^p, \frac{z_2}{z_1}] \oplus \mathbb{C}[\frac{z_1}{z_2}, z_2^p] \quad (\text{B.14})$$

and

$$w_1 = z_1^p \quad w_2 = \frac{z_2}{z_1} \quad (\text{B.15})$$

This is precisely the result we found in (4.9) for the $O_{\mathbb{P}_1}(-p)$ polynomial ring.

$\mathbb{C}^2/\Gamma_{5,2}$ -singularity

The cone associated to the resolved variety $\widehat{\mathbb{C}^2/\Gamma_{5,2}}$ is made out of three cones

$$\begin{aligned} \sigma_1 &= \{(0, 1), (1, 0)\} & \sigma_2 &= \{(1, 0), (3, -1)\} & \sigma_3 &= \{(3, -1), (5, -2)\} \\ \sigma_1^* &= \{(0, 1), (1, 0)\} & \sigma_2^* &= \{(0, -1), (1, 3)\} & \sigma_3^* &= \{(-1, -3), (2, 5)\} \end{aligned} \quad (\text{B.16})$$

This corresponds to blow up of two \mathbb{P}_1 's corresponding to the extra $v_2 = (0, 1)$ and $v_3 = (3, -1)$. The cone σ_N^* is made out of 3 cones with polynomial ring

$$\mathbb{C}[w_1, w_2] \oplus \mathbb{C}[w_2^{-1}, w_1 w_2^3] + \mathbb{C}[w_1^{-1} w_2^{-3}, w_1^2 w_2^5] = \mathbb{C}[z_1^5, \frac{z_2}{z_1^2}] \oplus \mathbb{C}[\frac{z_1^2}{z_2}, \frac{z_2^3}{z_1}] \oplus \mathbb{C}[\frac{z_1}{z_2^3}, z_2^5]$$

⁶We relabel $v_1 = (p, -q)$ in (B.5) as v_{p-1} .

and

$$w_1 = z_1^5 \quad w_2 = \frac{z_2}{z_1^2} \quad (\text{B.17})$$

This is precisely the result we found in (4.16) for the polynomial ring.

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